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Transition to Fulde-Ferrel-Larkin-Ovchinnikov phases near the tricritical point: an analytical study

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Abstract. We explore analytically the nature of the transition to the Fulde-Ferrel-Larkin-Ovchinnikov superfluid phases in the vicinity of the tricritical point, where these phases begin to appear. We make use of an expansion of the free energy up to an overall sixth order, both in order parameter amplitude and in wavevector. We first explore the minimization of this free energy within a subspace, made of arbitrary superpositions of plane waves with wavevectors of different orientations but same modulus. We show that the standard second order FFLO phase transition is unstable and that a first order transition occurs at higher temperature. Within this subspace we prove that it is favorable to have a real order parameter and that, among these states, those with the smallest number of plane waves are preferred. This leads to an order parameter with a $\cos(\mathbf{q}_0 \cdot \mathbf{r})$ dependence, in agreement with preceding work. Finally we show that the order parameter at the transition is only very slightly modified by higher harmonics contributions when the constraint of working within the above subspace is released.

PACS. 74.20.Fg BCS theory and its development – 74.60.Ec Mixed state, critical fields, and surface sheath

1 Introduction

Although they have been proposed a long time ago, Fulde-Ferrel-Larkin-Ovchinnikov (FFLO) phases [1,2] are still the subject of a continuing interest. Indeed the existence of these phases is a fairly remarkable phenomenon since they correspond to a spontaneous symmetry breaking of the standard BCS superfluid phase in the presence of an effective field, inducing a difference in chemical potential between the two populations involved in the formation of Cooper pairs. This symmetry breaking leads to an inhomogeneous superfluid with a space dependent order parameter, while the applied field is perfectly homogeneous. This situation is analogous to the appearance of vorticity in type II superconductors, but in this latter case the effect is due to the coupling of the field to particle currents while in FFLO phases only the coupling to the spin of the pairing fermions is involved. In standard superconductors the coupling to the orbital degrees of freedom is much stronger than the coupling to the spins. Hence the upper critical field is due to the orbital coupling and the FFLO phases can not be observed, since they should appear at much higher field. However in heavy fermions superconductors the strength of these two couplings is comparable, which could make possible the observation of FFLO phases. Nevertheless their sensitivity to impurities could be a major problem. Another possible direction to eliminate the orbital coupling is to consider lower dimensional superconductors, in a geometry where the currents would have to flow in an actually prohibited direction. Organic compounds or cuprate superconductors are interesting systems in this respect. And indeed very recently the FFLO state has been claimed to be observed in a quasi-two-dimensional organic compound [3]. On the other hand earlier possible observations in heavy fermion compounds [4] have not been undisputed. We note in particular that the analysis of experimental results relies very often heavily on the theoretical results, but we will see that the situation is not completely satisfactory in this respect.

Another class of physical systems where FFLO phases could be observed is coming up quite recently. These are the ultracold fermionic gases. As it is well known remarkably low temperatures have been obtained on bosonic gases, leading in particular to the observation of Bose-Einstein condensation in alkali ultracold gases. More recently fermionic gases have been cooled down in the degenerate regime [5–7] and reaching a BCS superfluid transition in these systems seems a reasonable possibility [8,9].

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However in the systems considered for observing this transition there is, in contrast to electronic spin relaxation in superconductors, no fast relaxation mechanism to equalize the populations of the two fermion species involved in the formation of Cooper pairs. Hence one should have no limitation to the effective field in these systems since the number of atoms in the populations can be in principle obtained at will. So the difference in atomic populations looks as a very promising control parameter. On the other hand if this parameter is not fully controlled this might very well be a major difficulty in reaching the BCS transition in these systems [10,11]. Let us also mention that FFLO phases are of high interest for quark matter [12] which is expected to be found in the core of compact stars.

In contrast with this raising experimental interest there are still theoretical problems with the precise nature of the possible phases. Specifically the basic FFLO instability corresponds to have pairs formed with a total nonzero momentum $\mathbf{q}_0$ instead of forming pairs $(\mathbf{k}, -\mathbf{k})$ with zero total momentum as in the standard BCS phase. This gives rise to a spatial dependence $\exp(\mathrm{i}\mathbf{q}_0 \cdot \mathbf{r})$ for the order parameter, which leaves a degeneracy with respect to the orientation of $\mathbf{q}_0$. This has been investigated by [2] Larkin and Ovchinnikov (LO) who looked how it is lifted right below the critical field. In this case, when considering the spatial dependence of the order parameter $\Delta(\mathbf{r})$, one can restrict the investigation to the subspace generated by linear combination of the plane waves $\exp(\mathrm{i}\mathbf{q}_0 \cdot \mathbf{r})$ with all possible directions $\mathbf{q}_0$. At $T = 0$, LO looked for periodic structures and found that the energetically favored result is a second order transition to a one-dimensional 'planar' texture $\Delta(\mathbf{r}) \sim \cos(\mathbf{q}_0 \cdot \mathbf{r})$. However they left open in their paper the possibility of a first order transition. Actually when considering the three-dimensional 'cubic' texture $\Delta(\mathbf{r}) \sim \cos(q_0 x) + \cos(q_0 y) + \cos(q_0 z)$ they found that it is energetically unfavorable compared to the normal state. Nevertheless they obtained from the gap equation that a nonzero solution for this order parameter exists *above* the FFLO transition line. In terms of the expansion of the free energy in powers of the order parameter, schematically $\Omega = \alpha_2 \Delta^2 + \alpha_4 \Delta^4$, this situation corresponds to a positive coefficient α_2 for the second order term and a negative coefficient α_4 for the fourth order one, just the opposite of the standard Landau-Ginzburg expansion below the transition. The LO evaluation of the related free energy corresponds actually to the maximum $\alpha_2^2/4|\alpha_4|$ of this free energy. Beyond this maximum the free energy decreases, and it would go to $-\infty$ if one would consider only the second and fourth order terms. Naturally one has to include the effect of all higher order terms in order to find the value of the free energy for large values of the order parameter. However, at the FFLO transition line and slightly above it, the free energy becomes negative for values of the order parameter (specifically for $\Delta^2 = \alpha_2/|\alpha_4|$) where only the second and fourth order terms have to be kept in the expansion. This shows definitely that the transition toward the FFLO phase occurs *above* the standard second order FFLO transition line and is actually first order, because one can display in this range a solution which has a lower free energy than the normal state. On the other hand in order to obtain consistently the order parameter which gives the lowest free energy one has naturally to take into account higher order terms. In particular it is by no means obvious that the cubic phase is the stable one. Moreover when higher order terms are considered there are no reason anymore to restrict the search to the subspace generated by the plane waves $\exp(\mathrm{i}\mathbf{q}_0 \cdot \mathbf{r})$.

In order to explore more fully this difficult problem with easier conditions, it is better to be able to proceed to some kind of expansion. This can be done if, instead of working at $T = 0$, one explores the vicinity of the tricritical point (TCP), where the FFLO transition line starts. It is located at $T_{\mathrm{tcp}}/T_{c0} = 0.561$ where T_{c0} is the critical temperature for $\bar{\mu} = 0$, with $2\bar{\mu} = \mu_\uparrow - \mu_\downarrow$ being half the chemical potential difference between the two fermionic populations forming pairs. The corresponding effective field is $\bar{\mu}_{\mathrm{tcp}}/T_{c0} = 1.073$. At this point, in the free energy expansion, both the coefficient of the second order term α_2 and the coefficient of the fourth order term α_4 vanish. Naturally α_2 is zero just because the TCP is on the standard second order phase transition line. On the other hand following this transition line one has $\alpha_4 > 0$ for $T > T_{\mathrm{tcp}}$ and $\alpha_4 < 0$ for $T < T_{\mathrm{tcp}}$. In our case the change of sign of α_4 at T_{tcp} is the origin of the FFLO instability, because it happens accidentally that this same coefficient controls also the wavevector dependence of α_2 (this is seen explicitely in Eqs. (7, 8) below). Clearly, by continuity, the equilibrium order parameter will be small in the vicinity of the TCP since we know that it is zero just above the TCP on the second order line and it is small right below it in the superfluid phase. Therefore a power expansion of the free energy will be enough to find it. On the other hand, since the second and fourth order terms are zero at the TCP, we have clearly to expand at least up to sixth order, but this will prove to be enough because the corresponding coefficient is positive and not small. Similarly the optimum wavevector $\mathbf{q}_0$ corresponding to the FFLO phase will be small in the vicinity of the TCP since it is zero just above it on the second order line. This allows to proceed to a gradient expansion of the free energy. Since only even powers of the wavevector can enter and we look for a minimum as a function of this wavevector, we have to expand at least up to fourth order in gradient, but this will prove again to be enough.

This line of thought has actually already been followed by Houzet *et al.* [13,16], who have performed this expansion for the free energy and explored the result numerically. They have found that the energetically favored phase is the one-dimensional planar order parameter found by LO at $T = 0$, but that the transition is actually slightly first order, instead of second order as found by LO at $T = 0$. Our purpose in the present paper is rather to proceed to an analytical study of this problem. Indeed there are infinitely many possible order parameters in competition. And our aim, in considering the vicinity of the TCP, is to find the important ingredients which are responsible for the selection of the actual stable state and obtain a better physical understanding, having in particular in mind the generalization to more complicated situations. Hence our paper is complementary to their work. In particular we obtain a first order transition to the one-dimensional planar order parameter, but we will be able to analyze the reasons which favor this phase. The transition to this planar order parameter has been actually explored numerically down to $T = 0$ by Matsuo *et al.* [14]. They have used quasiclassical equations and found that the transition keeps first order down to low temperature, but eventually goes to second order in agreement with LO. On the other hand since we know that at $T = 0$ the cubic phase is more stable than the planar one, the question of the stablest phase at low temperature is still unsolved. Finally this first order transition to the planar phase in the three-dimensional case is in contrast with the results of Burkhardt and Rainer [15] who found it to be second order in a two-dimensional space.

In the following section, for completeness and to set up our notations, we rederive the expression [16] of the free energy. After considering in Section 3 some simple situations, we explore in details the minimization of this free energy. This is done in Section 4 by restricting our search to the LO subspace for the order parameter, which is made of arbitrary superpositions of plane waves. In Section 5 we show that our results are only slightly modified when we release this restriction. Throughout the paper we restrict ourselves to the simplest BCS scheme, namely we will consider the free energy corresponding to a weak coupling isotropic Fermi system, ignoring in particular any Fermi liquid effect. Moreover we concentrate on the three-dimensional case which leads to a first order transition, and make only occasionally comparison with the two-dimensional situation where the transition is second order.

2 The free energy

There are various ways to obtain the explicit expression for the free energy we need in the vicinity of the TCP [17,18]. In practice it is convenient to use the fact [18] that, by varying the free energy with respect to $\Delta(\mathbf{r})$ one finds the gap equation, which is also easily obtained from Gorkovs equations, as done for example by LO [2]. The integral form of these equations is [17,19], with standard notations:

$$G_\uparrow(\mathbf{r},\mathbf{r}') = G^0_\uparrow(\mathbf{r}-\mathbf{r}') - \int \mathrm{d}\mathbf{r}_1 G^0_\uparrow(\mathbf{r}-\mathbf{r}_1)\Delta(\mathbf{r}_1)F^+(\mathbf{r}_1,\mathbf{r}') \tag{1}$$

$$F^+(\mathbf{r},\mathbf{r}') = \int \mathrm{d}\mathbf{r}_1 \bar{G}^0_\downarrow(\mathbf{r}-\mathbf{r}_1)\Delta^*(\mathbf{r}_1)G_\uparrow(\mathbf{r}_1,\mathbf{r}') \tag{2}$$

with, for the Fourier transforms of the free fermions thermal propagators, $G^0_\uparrow(\mathbf{k}) = (\mathrm{i}\omega_n - \xi_\mathbf{k} + \bar{\mu})^{-1}$ and $\bar{G}^0_\downarrow(\mathbf{k}) = (-\mathrm{i}\omega_n - \xi_k - \bar{\mu})^{-1}$ where $\xi_\mathbf{k}$ is the kinetic energy measured from the Fermi surface for $\bar{\mu} = 0$ and $\omega_n = \pi T(2n+1)$ are Matsubara frequencies. The order parameter is given by the self-consistency relation:

$$\Delta^*(\mathbf{r}) = VT\sum_n F^+(\mathbf{r},\mathbf{r}). \tag{3}$$

We expand equations (1–2) up to fifth order in $\Delta(\mathbf{r})$. We introduce the Fourier transform $\Delta_\mathbf{q} = \int \mathrm{d}\mathbf{r}\Delta(\mathbf{r})\exp(-\mathrm{i}\mathbf{q}\cdot\mathbf{r})$ of the order parameter. As explained in the introduction we proceed also to an expansion in the wavevector $\mathbf{q}$ of the order parameter since we know that its relevant values will be small in the vicinity of the TCP. More precisely we will see that, in order to obtain a coherent expansion, it is enough to go only up to fifth order terms in overall power of Δ and $\mathbf{q}$. This means that, in the gap equation, we have to expand the first order term in Δ only to fourth order in $\mathbf{q}$. Similarly the third order term in Δ has to be expanded only to second order in $\mathbf{q}$ and the fifth order term in Δ can be calculated to zeroth order in $\mathbf{q}$. For example in order to find the third order term, we have to expand up to second order in wavevectors:

$$\sum_k \bar{G}^0(\mathbf{k})G^0(\mathbf{k}+\mathbf{q}_1)\bar{G}^0(\mathbf{k}+\mathbf{q}-\mathbf{q}_3)G^0(\mathbf{k}+\mathbf{q})\Delta^*_{\mathbf{q}_1}\Delta_{\mathbf{q}_2}\Delta^*_{\mathbf{q}_3} \tag{4}$$

where we have used $\mathbf{q}_1+\mathbf{q}_3 = \mathbf{q}+\mathbf{q}_2$ and we have omitted the unnecessary spin index. In the expansion appear the following numerical coefficients:

$$a_0(\bar{\mu},T) = \frac{1}{N_0V} - 2\pi T\,\mathrm{Re}\left[\sum_{n=0}^{\infty}\frac{1}{\bar{\omega}_n}\right]$$

$$a_2(\bar{\mu}/T) = -\bar{\mu}^2 2\pi T\,\mathrm{Re}\left[\sum_{n=0}^{\infty}\frac{1}{\bar{\omega}_n^3}\right] \tag{5}$$

$$a_4(\bar{\mu}/T) = -\bar{\mu}^4 2\pi T\,\mathrm{Re}\left[\sum_{n=0}^{\infty}\frac{1}{\bar{\omega}_n^5}\right]$$

where $\bar{\omega}_n = \omega_n - \mathrm{i}\bar{\mu}$, and the summation for a_0 has to be cut-off in the standard BCS way. The simple second order transition line to a standard BCS superfluid with space independent order parameter is given by $a_0(\bar{\mu},T) = 0$. Below the TCP it corresponds to a spinodal transition line, at which the normal state becomes absolutely unstable against a transition toward a space independent order parameter. The domain $a_0(\bar{\mu},T) > 0$ corresponds to the region of the $(\bar{\mu},T)$ phase diagram above this line, and it is the domain where we will look for other transitions. In practice we can see $a_0(\bar{\mu},T)$ as a measure of the distance from the spinodal line in the $(\bar{\mu},T)$ plane. Explicitely if we define $T_{sp}(\bar{\mu}/T)$ the spinodal temperature as a function of the ratio $\bar{\mu}/T$, we have $a_0(\bar{\mu},T) = \ln[T/T_{sp}(\bar{\mu}/T)]$. We will not need to explicit further this distance. As indicated in the introduction we have by definition $a_2(\bar{\mu}/T) = 0$ at the TCP and it is small in the vicinity of this point. For $(\bar{\mu}/T) > (\bar{\mu}/T)_{\mathrm{tcp}} = 1.91$, we have $a_2 > 0$ and $a_2 < 0$ for $(\bar{\mu}/T) < (\bar{\mu}/T)_{\mathrm{tcp}}$. Finally $a_4(\bar{\mu}/T) = 0.114$ at the TCP (while it is negative near $\bar{\mu} = 0$ and goes to -0.25 when

$T \to 0$). With these notations the gap equation in the vicinity of the TCP reads:

$$\Delta_{\mathbf{q}}\left[a_0-\frac{1}{3}a_2Q^2+\frac{1}{5}a_4Q^4\right]-\sum_{\mathbf{q}_i}\Delta_{\mathbf{q}_1}\Delta^*_{\mathbf{q}_2}\Delta_{\mathbf{q}_3}$$
$$\times\left[\frac{1}{2}a_2-\frac{1}{6}a_4\left(Q^2+2Q_1^2+2Q_3^2-\mathbf{Q}\cdot(\mathbf{Q}_1+\mathbf{Q}_3)+3\mathbf{Q}_1\cdot\mathbf{Q}_3\right)\right]$$
$$+\frac{3}{8}a_4\sum_{\mathbf{q}_i}\Delta_{\mathbf{q}_1}\Delta^*_{\mathbf{q}_2}\Delta_{\mathbf{q}_3}\Delta^*_{\mathbf{q}_4}\Delta_{\mathbf{q}_5}=0 \quad (6)$$

where we have used the dimensionless wavevector $\mathbf{Q} = \mathbf{q}v_F/2\bar{\mu}$ and expressed $\Delta_{\mathbf{q}}$ in units of $\bar{\mu}$. Also the momentum conservation is assumed in the summations, that is $\mathbf{q}_1+\mathbf{q}_3 = \mathbf{q}+\mathbf{q}_2$ in the third order term and $\mathbf{q}_1+\mathbf{q}_3+\mathbf{q}_5 = \mathbf{q}+\mathbf{q}_2+\mathbf{q}_4$ in the fifth order one. The above expression can be checked against the case of the simple Fulde-Ferrell state $\Delta(\mathbf{r}) = \exp(i\mathbf{q}_0\cdot\mathbf{r})$ where a single wavevector enters.

Now the above gap equation is obtained by minimizing the following free energy difference Ω between the superfluid and the normal state:

$$\Omega=\sum_{\mathbf{q}}|\Delta_{\mathbf{q}}|^2\left[a_0-\frac{1}{3}a_2Q^2+\frac{1}{5}a_4Q^4\right]-\frac{1}{2}\sum_{\mathbf{q}_i}\Delta_{\mathbf{q}_1}\Delta^*_{\mathbf{q}_2}\Delta_{\mathbf{q}_3}\Delta^*_{\mathbf{q}_4}$$
$$\times\left[\frac{1}{2}a_2-\frac{1}{6}a_4(Q_1^2+Q_2^2+Q_3^2+Q_4^2+\mathbf{Q}_1\cdot\mathbf{Q}_3\right.$$
$$\left.+\mathbf{Q}_2\cdot\mathbf{Q}_4)\right]+\frac{1}{8}a_4\sum_{\mathbf{q}_i}\Delta_{\mathbf{q}_1}\Delta^*_{\mathbf{q}_2}\Delta_{\mathbf{q}_3}\Delta^*_{\mathbf{q}_4}\Delta_{\mathbf{q}_5}\Delta^*_{\mathbf{q}_6} \quad (7)$$

where we have the momentum conservation $\mathbf{q}_1+\mathbf{q}_3 = \mathbf{q}_2+\mathbf{q}_4$ in the fourth order term while $\mathbf{q}_1+\mathbf{q}_3+\mathbf{q}_5 = \mathbf{q}_2+\mathbf{q}_4+\mathbf{q}_6$ holds in the sixth order one. We have used symmetry and momentum conservation to present the fourth order term in a symmetrical way. This expression equation (7) is just the free energy we were looking for. It coincides exactly with the result of reference [16] once the differences in notations are taken into account. We have considered here the 3-D case. For a two-dimensional system the angular averages found in the calculation are different. The result is simply obtained from the above one by multiplying the Q^2 terms by 3/2 and the Q^4 terms by 15/8.

3 Simple cases

Let us first consider some simple situations. If we consider an homogeneous order parameter, that is $\mathbf{q} = 0$, we have merely $\Omega = a_0\Delta^2 - a_2\Delta^4/4 + a_4\Delta^6/8$. If we want to have this free energy negative for $a_0 > 0$ we need to have $a_2 > 0$, that is to be at temperature below the TCP. In this case $\Omega > 0$ when $a_0 > a_2^2/8a_4$ and we reach a first order transition for $a_0 = a_2^2/8a_4$, with a non zero order parameter $\Delta^2 = a_2/a_4$. This is the standard first order Pauli limiting transition. We consider next the possibility of a second order transition. In this case only the second order term in equation (7) is relevant. The location of this transition is given by $a_0 = \frac{1}{3}a_2Q^2 - \frac{1}{5}a_4Q^4$. Below the TCP, where $a_2 > 0$, we can find $a_0 > 0$ for non zero wavevector $\mathbf{Q}$, that is we will find an FFLO phase. Precisely the optimal wavevector is $Q_0^2 = \frac{5}{6}a_2/a_4$ and the corresponding maximal a_0 is $a_0 = \frac{5}{36}a_2^2/a_4$. We see that this value is larger than the one we just found for the standard Pauli limiting transition. Thus as expected the FFLO transition happens first and overtakes the first order transition. Finally it is natural and interesting to try to generalize the two above situations and consider the possibility of a first order transition for an order parameter with a single wavector component $\Delta_{\mathbf{q}}$. With the shorthand $\Delta_{\mathbf{q}} \equiv \Delta$ the free energy writes:

$$\Omega=\left[a_0-\frac{1}{3}a_2Q^2+\frac{1}{5}a_4Q^4\right]\Delta^2-\frac{1}{4}\left[a_2-2a_4Q^2\right]\Delta^4+\frac{1}{8}a_4\Delta^6. \quad (8)$$

Minimizing first with respect to Q^2 we obtain for the extremum the condition $Q^2 = \frac{5}{6}a_2/a_4 - \frac{5}{4}\Delta^2$, which implies that must have $\Delta^2 \le \frac{2}{3}a_2/a_4$ otherwise we are back to the homogeneous situation and the Pauli limiting transition. Inserting this value for Q^2 in equation (8) we find $\Omega = \left[a_0 - \frac{5}{36}a_2^2/a_4\right]\Delta^2 + \frac{1}{6}a_2\Delta^4 - \frac{3}{16}a_4\Delta^6$. We are naturally interested in finding a transition higher than the standard FFLO. This means we are looking for $a_0 > \frac{5}{36}a_2^2/a_4$, so the first term in Ω is positive. But one sees that the sum of the last two terms is also positive for $\Delta^2 \le \frac{2}{3}a_2/a_4$. Therefore we have not been able to improve the standard FFLO solution. However we have clearly not done our best in this direction.

Before trying to improve in this way, it is convenient to simplify our expression for the free energy by taking reduced units for the order parameter and the wavector, which come out naturally from our above discussion. We set $\Delta = (a_2/a_4)^{1/2}\bar{\Delta}$, $\mathbf{Q} = (a_2/a_4)^{1/2}\bar{\mathbf{q}}$, $a_0 = A_0a_2^2/a_4$ and $\Omega = (a_2^3/a_4^2)F$. This leads to rewrite equation (7) for the free energy as:

$$F=\sum_{\mathbf{q}}|\bar{\Delta}_{\mathbf{q}}|^2\left[A_0-\frac{1}{3}\bar{q}^2+\frac{1}{5}\bar{q}^4\right]-\sum_{\mathbf{q}_i}\bar{\Delta}_{\mathbf{q}_1}\bar{\Delta}^*_{\mathbf{q}_2}\bar{\Delta}_{\mathbf{q}_3}\bar{\Delta}^*_{\mathbf{q}_4}$$
$$\times\left[\frac{1}{4}-\frac{1}{12}(\bar{q}_1^2+\bar{q}_2^2+\bar{q}_3^2+\bar{q}_4^2+\bar{\mathbf{q}}_1\cdot\bar{\mathbf{q}}_3+\bar{\mathbf{q}}_2\cdot\bar{\mathbf{q}}_4)\right]$$
$$+\frac{1}{8}\sum_{\mathbf{q}_i}\bar{\Delta}_{\mathbf{q}_1}\bar{\Delta}^*_{\mathbf{q}_2}\bar{\Delta}_{\mathbf{q}_3}\bar{\Delta}^*_{\mathbf{q}_4}\bar{\Delta}_{\mathbf{q}_5}\bar{\Delta}^*_{\mathbf{q}_6}. \quad (9)$$

It is clear from this rescaling transformation and the resulting expression that, in the vicinity of the TCP (where a_2 is small), it is unnecessary to go beyond our sixth order expansion in Δ and $\mathbf{q}$. It is also of interest to rewrite this free energy as a functional of $\Delta(\mathbf{r})$ by Fourier transform. This gives, after by parts integrations:

$$F=\int d\mathbf{r}\left[A_0|\bar{\Delta}|^2-\frac{1}{3}|\nabla\bar{\Delta}|^2+\frac{1}{5}|\nabla^2\bar{\Delta}|^2\right]$$
$$-\int d\mathbf{r}\left[\frac{1}{4}|\bar{\Delta}|^4-\frac{1}{24}\left[2\left(\nabla|\bar{\Delta}|^2\right)^2\right.\right.$$
$$\left.\left.+3\left(\nabla\bar{\Delta}^2\right)\left(\nabla\bar{\Delta}^{*2}\right)\right]\right]+\frac{1}{8}\int d\mathbf{r}|\bar{\Delta}|^6. \quad (10)$$

4 The LO subspace

As emphasized by LO all the states corresponding to the same wavector $\bar{q}_0$ but with different orientation for $\bar{\mathbf{q}}_0$ are degenerate right on the FFLO transition line. This degeneracy is lifted, at least partially, when one goes into the superfluid phase because of the coupling between the various plane waves produced by the nonlinear terms in the free energy. When one investigates the states selected in this process, one has to consider the subspace:

$$\bar{\Delta}(\mathbf{r}) = \sum \Delta_{\mathbf{q}_0} \exp(\mathrm{i}\bar{\mathbf{q}}_0 \cdot \mathbf{r}) \quad (11)$$

of all the order parameters generated by these plane waves. We call this the LO subspace. In this section we will restrict to this subspace our search for the state appearing at the transition: we will look for the minimum of the free energy within this LO subspace. Actually LO looked for a lattice as a solution and restricted themselves to this kind of order parameter. However there is physically no basic reason to enforce this type of restriction. One could look for incommensurate structures or quasicrystal-like solutions. Even if these are not the lowest energy solution, they might be of interest as local solutions corresponding physically to defects. Therefore we have not set a periodicity condition on the solutions we have considered. Nevertheless let us indicate at once that the energetically favored solutions we have found within the LO subspace are actually periodic. Although considering only the LO subspace is an important restriction, it does not make the problem easy at all, although we solve it completely below.

Let us first show that, within this subspace, the standard FFLO transition line is not stable, and the transition is actually first order. With our reduced units the LO subspace corresponds to $\bar{\mathbf{q}}_0^2 = \frac{5}{6}$. This minimizes the coefficient of the second order term in equation (9). The FFLO transition line is then given by $A_0 = \frac{5}{36}$, which makes the second order term zero within the LO subspace. Let us then look at the fourth order term in equation (9). This amounts to calculate the functional derivative of the free energy on the FFLO transition line. In a standard second order phase transition it should always be positive, forcing the order parameter to be zero on the transition line. However in the present case it is not obvious that this is systematically so because of the interplay of the wavevectors in this term. Specifically we introduce a parameter β which describes this effect for any fixed order parameter $\Delta(\mathbf{r})$. It is defined by:

$$2\beta\bar{q}_0^2 \sum_{\mathbf{q}_i} \bar{\Delta}_{\mathbf{q}_1} \bar{\Delta}^*_{\mathbf{q}_2} \bar{\Delta}_{\mathbf{q}_3} \bar{\Delta}^*_{\mathbf{q}_4} = \bar{q}_0^2 \sum_{\mathbf{q}_i} (\hat{\mathbf{q}}_1 \cdot \hat{\mathbf{q}}_3 + \hat{\mathbf{q}}_2 \cdot \hat{\mathbf{q}}_4) \bar{\Delta}_{\mathbf{q}_1} \bar{\Delta}^*_{\mathbf{q}_2} \bar{\Delta}_{\mathbf{q}_3} \bar{\Delta}^*_{\mathbf{q}_4} = - \int \bar{\Delta}^2 (\nabla \bar{\Delta}^*)^2 + \text{c.c.} \quad (12)$$

For the simple case of the FF solution we have merely $\beta = 1$. However this is the highest possible value and we can think of decreasing it, or even making it negative, by a proper choice of the order parameter in the LO space, although naturally $\beta \geq -1$. Then the fourth order term in equation (9) is given by $\left[\frac{1}{6}(\beta+2)\bar{q}_0^2 - \frac{1}{4}\right] \int |\bar{\Delta}|^4$ and has thus the same sign as $5\beta + 1$ when we take $\bar{q}_0^2 = \frac{5}{6}$. Hence for any order parameter with $\beta \leq -\frac{1}{5}$ the fourth order term is negative. We will find actually many such states. Now when we are on the FFLO transition line (the second order term is zero) and the fourth order term is negative, we decrease the free energy and make it negative just by taking the order parameter to be small and nonzero, which shows that the standard FFLO transition line is unstable. Naturally the larger the order parameter, the lower the free energy, but we have to stay in the range where the second and fourth order terms are the only ones important in our expansion, which means that the sixth order term is negligible. Then since we have a negative free energy, we can raise it back to zero by increasing A_0 beyond its value $\frac{5}{36}$ on the FFLO line, which means we go beyond this line in the $(\bar{\mu}, T)$ phase diagram. In this way the second order term becomes positive. Eventually we will be limited by the growth of the sixth order term. The transition we have found corresponds to positive second and sixth order terms and a negative fourth order one. The free energy becomes negative for a nonzero value of the order parameter. We have thus found a first order transition beyond the standard FFLO transition line. This discussion about the order of the transition is the exact analogue of the one we made in the introduction for the $T = 0$ situation.

Naturally it is of interest to minimize β since it is rather natural to expect that the states corresponding to the minimum will lead to the stronger instability toward the first order transition. We show now that $\beta \geq -\frac{1}{3}$, the equality $\beta = -\frac{1}{3}$ being obtained for *any* real order parameter. We make use of:

$$\int \left[\bar{\Delta}^2(\nabla\bar{\Delta}^*)^2 - |\bar{\Delta}|^2|\nabla\bar{\Delta}|^2\right] + \text{c.c.} = \int \left[\bar{\Delta}\nabla\bar{\Delta}^* - \text{c.c.}\right]^2 \leq 0 \quad (13)$$

where the equality occurs only for a real order parameter (within an irrelevant overall constant phase factor), and:

$$\int \left[\bar{\Delta}^2(\nabla\bar{\Delta}^*)^2 + 2|\bar{\Delta}|^2|\nabla\bar{\Delta}|^2\right] + \text{c.c.} = - \int |\bar{\Delta}|^2(\bar{\Delta}\nabla^2\bar{\Delta}^* + \text{c.c.}) = 2\bar{q}_0^2 \int |\bar{\Delta}|^4 \quad (14)$$

where the last step makes specific use of the form equation (11) for the order parameter. We have then:

$$- 2\beta\bar{q}_0^2 \int |\bar{\Delta}|^4 = \int \bar{\Delta}^2(\nabla\bar{\Delta}^*)^2 + \text{c.c.} \leq \int \Big[\bar{\Delta}^2(\nabla\bar{\Delta}^*)^2 - \frac{2}{3}\left[\bar{\Delta}^2(\nabla\bar{\Delta}^*)^2 - |\bar{\Delta}|^2|\nabla\bar{\Delta}|^2\right]\Big] + \text{c.c.} = \frac{1}{3}\int [\bar{\Delta}^2(\nabla\bar{\Delta}^*)^2 + 2|\bar{\Delta}|^2|\nabla\bar{\Delta}|^2] + \text{c.c.} = \frac{2}{3}\bar{q}_0^2 \int |\bar{\Delta}|^4 \quad (15)$$

hence $\beta \geq -\frac{1}{3}$.

Together with β it is also intuitively convenient to consider γ defined by:

$$\gamma \int d\mathbf{r}|\bar{\Delta}|^4 = \bar{q}_0^{-2} \int d\mathbf{r} \left[\nabla|\bar{\Delta}|^2\right]^2 = \sum_{\mathbf{q}_i} (\hat{\mathbf{q}}_1 - \hat{\mathbf{q}}_2)^2 \bar{\Delta}_{\mathbf{q}_1} \bar{\Delta}^*_{\mathbf{q}_2} \bar{\Delta}_{\mathbf{q}_3} \bar{\Delta}^*_{\mathbf{q}_4} \quad (16)$$

which is easily seen to satisfy $\gamma = 1-\beta$ from equation (14). We are thus interested in maximizing γ. From the first expression in equation (16) it is intuitively clear that γ will be large when the (properly normalized) order parameter has strong spatial variations. In particular it will be better for $|\bar{\Delta}(\mathbf{r})|^2$ to have many nodes. This is more easily achieved if $\bar{\Delta}(\mathbf{r})$ has no imaginary part since in this case one has only to require that the real part is zero. This makes intuitively reasonable that γ is maximized by a real order parameter. One can also come to this conclusion from the Fourier expansion in equation (16) (or also from Eq. (12)): it is of interest to have as often as possible opposite wavevectors so that $(\hat{\mathbf{q}}_1 - \hat{\mathbf{q}}_2)^2$ takes as much as possible its maximum value, namely 4. One is thus naturally led to an order parameter which is a combination of $\cos(\mathbf{q}_0 \cdot \mathbf{r} + \varphi_{\mathbf{q}_0})$ with real coefficients, the directions of the $\mathbf{q}_0$'s being free. This corresponds merely to require that $\bar{\Delta}(\mathbf{r})$ is real in equation (11) by taking $\Delta_{-\mathbf{q}_0} = \Delta^*_{\mathbf{q}_0}$. On the other hand, since any real order parameter gives the maximum γ, it is not necessary that these cosines have equal weight.

We can now come back to our free energy equation (9) and find the best solution within the LO subspace, since we have found that the minimum β is $-1/3$, as soon as the order parameter is real, which implies $\Delta_{-\mathbf{q}_0} = \Delta^*_{\mathbf{q}_0}$ in equation (11). We introduce a measure $\bar{\Delta}$ of the amplitude of the order parameter by setting:

$$\int d\mathbf{r}|\bar{\Delta}|^2 = \sum_{\mathbf{q}_i} \bar{\Delta}_{\mathbf{q}_1} \bar{\Delta}^*_{\mathbf{q}_2} = N_2 \bar{\Delta}^2 \quad (17)$$

where by definition $N_2 \equiv N$ is the number of plane waves coming in equation (11). If all the planed waves have same amplitude, $\bar{\Delta}$ is just the common value of these amplitudes. Then we define N_4 and N_6 by:

$$\int |\bar{\Delta}|^4 = \sum_{\mathbf{q}_i} \bar{\Delta}_{\mathbf{q}_1} \bar{\Delta}^*_{\mathbf{q}_2} \bar{\Delta}_{\mathbf{q}_3} \bar{\Delta}^*_{\mathbf{q}_4} = N_4 \bar{\Delta}^4 \quad (18)$$

and:

$$\int |\bar{\Delta}|^6 = \sum_{\mathbf{q}_i} \bar{\Delta}_{\mathbf{q}_1} \bar{\Delta}^*_{\mathbf{q}_2} \bar{\Delta}_{\mathbf{q}_3} \bar{\Delta}^*_{\mathbf{q}_4} \bar{\Delta}_{\mathbf{q}_5} \bar{\Delta}^*_{\mathbf{q}_6} = N_6 \bar{\Delta}^6. \quad (19)$$

In these definitions we have used the same implicit convention of momentum conservation as in equation (6), that is $\mathbf{q}_1 = \mathbf{q}_2$ in equation (17), $\mathbf{q}_1 + \mathbf{q}_3 = \mathbf{q}_2 + \mathbf{q}_4$ in equation (18) and $\mathbf{q}_1 + \mathbf{q}_3 + \mathbf{q}_5 = \mathbf{q}_2 + \mathbf{q}_4 + \mathbf{q}_6$ in equation (19). For the simple plane wave considered in equation (8), we had $N_2 = N_4 = N_6 = 1$. For the real order parameter we are interested in, the set of wavevectors $\{\mathbf{q}_i\}$ is made of $N/2$ pairs. If all the planed waves have same amplitude, one finds [20] for example by simple counting that $N_4 = 3N(N-1)$ and $N_6 = 5N(3N^2 - 9N + 8)$. Actually, once it is recognized that, from the counting procedure, N_4 and N_6 are polynomials in N of order 2 and 3 respectively, the coefficient may easily be found by considering the cases $N = 2, 4, 6$. With these notations the free energy equation (9) reduces to:

$$F = N_2 \bar{\Delta}^2 \left[A_0 - \frac{1}{3}\bar{q}^2 + \frac{1}{5}\bar{q}^4\right] - N_4 \bar{\Delta}^4 \left[\frac{1}{4} - \frac{\alpha}{2}\bar{q}^2\right] + \frac{1}{8} N_6 \bar{\Delta}^6 \quad (20)$$

where we have set $\alpha = \frac{\beta+2}{3}$ (for the simple plane wave considered in equation (8), we had $\alpha = 1$).

We proceed now as we have done for equation (8). Minimizing F with respect to $\bar{q}^2$ we find for the extremum the condition $\bar{q}^2 = \frac{5}{6} - \frac{5}{4}\alpha(N_4/N_2)\bar{\Delta}^2$, which implies that, in our considerations, we have for $\bar{\Delta}^2$ an upper bound $\bar{\Delta}^2_{max} = (2/3\alpha)N_2/N_4$. This leads, for the value of the free energy F at this extremum, to:

$$\frac{F}{\bar{\Delta}^2} = N_2 \left(A_0 - \frac{5}{36}\right) + N_4 \bar{\Delta}^2 \left(\frac{5\alpha}{12} - \frac{1}{4}\right) + \bar{\Delta}^4 \left(\frac{N_6}{8} - \frac{5\alpha^2}{16}\frac{N_4^2}{N_2}\right) \cdot \quad (21)$$

Now the standard FFLO solution corresponds to $A_0 = \frac{5}{36}$. Since we are interested in a better solution we want $A_0 > \frac{5}{36}$ which makes the first term of $F/\bar{\Delta}^2$ positive. On the other hand, for $\bar{\Delta} = \bar{\Delta}_{max}$, the sum of the last two terms in the right hand side (r.h.s.) of equation (21) can be written as $\bar{\Delta}^2_{max}(N_4/24\alpha)[5(\alpha - \frac{3}{5})^2 + 2N_2N_6/N_4^2 - \frac{9}{5}]$. This is always positive since we have $N_2N_6/N_4^2 \geq 1$ (this results directly from $\int d\mathbf{r}|\bar{\Delta}|^2 \cdot \int d\mathbf{r}|\bar{\Delta}|^6 \geq [\int d\mathbf{r}|\bar{\Delta}|^4]^2$). If we assume that the second term in the r.h.s. of equation (21) is positive, this implies that the free energy is always positive. Therefore if we want to find non positive values for F it is necessary to have a negative coefficient for the second term in the r.h.s. of equation (21), which means $\alpha < \frac{3}{5}$. This is possible since the minimum $\beta = -\frac{1}{3}$ we have found above corresponds to a minimum $\alpha = \frac{5}{9}$.

Then the quadratic form equation (21) can be zero if we meet the condition $4(A_0 - \frac{5}{36})(2\frac{N_2N_6}{N_4^2} - 5\alpha^2) \leq (\frac{5\alpha}{3} - 1)^2$. This leads to the following result for the transition line to the FFLO phase:

$$A_0 = \frac{5}{36} + \frac{1}{8} \frac{\left(1 - \frac{5\alpha}{3}\right)^2}{\frac{N_2N_6}{N_4^2} - \frac{5}{2}\alpha^2} \cdot \quad (22)$$

In the $(\bar{\mu} - T)$ plane this line is higher than the standard second order FFLO transition line, which is given by $A_0 = \frac{5}{36}$. On the other hand the value of $\bar{\Delta}_m$ which gives $F = 0$ at the threshold given by equation (21) is:

$$\bar{\Delta}^2_m = \frac{N_2}{N_4} \frac{1 - \frac{5\alpha}{3}}{\frac{N_2N_6}{N_4^2} - \frac{5}{2}\alpha^2} \quad (23)$$

which is clearly nonzero and our transition is quite explicitely a first order transition (note that the condition $\bar{\Delta}_m < \bar{\Delta}_{max}$ is necessarily satisfied since F is zero for $\bar{\Delta}_m$ and positive for $\bar{\Delta}_{max}$).

Coming back to the location of the transition line equation (23) we consider now how to optimize it. First we see from the numerator that it is advantageous to lower α as much as possible. Indeed our minimum $\alpha = \frac{5}{9}$ is quite close to the limiting value $\alpha = \frac{3}{5}$ so the variation of the denominator with α is irrelevant. Hence we are lead to make the fourth order term in the free energy equation (9) as negative as possible by minimizing β, as we anticipated at the beginning of Section 4. Once we have taken $\alpha = \frac{5}{9}$, we see that it is of interest to take N_2N_6/N_4^2 as small as possible (we have seen that it is bounded from below by 1). From their definitions equations (18, 19) we can evaluate N_4 and N_6 for a general order parameter equation (11) $\bar{\Delta}(\mathbf{r}) = \sum \Delta_{\mathbf{q}_i} \exp(\mathrm{i}\mathbf{q}_i \cdot \mathbf{r}) = 2\sum |\Delta_{\mathbf{q}_i}| \cos(\mathbf{q}_i \cdot \mathbf{r} + \varphi_{\mathbf{q}_i})$. One finds:

$$N_4\bar{\Delta}^4 = 3\left(\sum_i |\Delta_{\mathbf{q}_i}|^2\right)^2 - 3\sum_i |\Delta_{\mathbf{q}_i}|^4 \quad (24)$$

and:

$$N_6\bar{\Delta}^6 = 15\left(\sum_i |\Delta_{\mathbf{q}_i}|^2\right)^3 - 45\left(\sum_i |\Delta_{\mathbf{q}_i}|^2\right)^2 \sum_i |\Delta_{\mathbf{q}_i}|^4 + 40\sum_i |\Delta_{\mathbf{q}_i}|^6. \quad (25)$$

Actually one can recognize that, for symmetry reasons, the results involve only $\sum_i |\Delta_{\mathbf{q}_i}|^n$ with $n = 2, 4, 6$, because odd powers of cosines average to zero. Hence the results assume necessarily the general form given by equations (24, 25). Then the coefficients are easily obtained from the specific case where all the amplitudes $|\Delta_{\mathbf{q}_i}|$ are equal.

Since the wavevectors are paired it is now more convenient to sum over pairs from now on. Defining $a_i = |\Delta_{\mathbf{q}_i}|^2/\sum_i |\Delta_{\mathbf{q}_i}|^2$ (implying $\sum_i a_i = 1$) we have:

$$\frac{9}{10} N_2N_6/N_4^2 = \frac{6 - 9S_2 + 4S_3}{(2 - S_2)^2} \quad (26)$$

where $S_2 = \sum_i a_i^2$ and $S_3 = \sum_i a_i^3$. When we have a single pair $S_2 = S_3 = 1$ and the r.h.s. of equation (26) is equal to 1. We show now that it is otherwise larger than 1. Since $(2 - S_2)^2 < 4 - 3S_2$ because $S_2 < 1$ when we have more than a single pair, it is enough to prove that $3(1 - S_2) \geq 2(1 - S_3)$. This is in turn verified because we can write for the left-hand side $1 - S_2 = (\sum_i a_i)^2 - \sum_i a_i^2 = 2\sum_{i<j} a_i a_j$. In a similar way we have in the right-hand side $1 - S_3 = (\sum_i a_i)^3 - \sum_i a_i^3 = 3\sum_{i<j} a_i a_j (a_i + a_j) + 6\sum_{i<j<k} a_i a_j a_k = 3\sum_{i<j} a_i a_j - 9\sum_{i<j<k} a_i a_j a_k$. So our statement is correct since it is just equivalent to $\sum_{i<j<k} a_i a_j a_k \geq 0$ (the equality holds when we have only two pairs since one can not have three different indices). Therefore we come to the conclusion that N_2N_6/N_4^2 is minimized when we take a single pair of plane waves with wavevectors $(\mathbf{q}, -\mathbf{q})$, corresponding to a simple order parameter proportional to $\cos(\mathbf{q} \cdot \mathbf{r})$, which has hence a planar symmetry. It is unfavourable to increase the number of plane waves. This is more easily seen in the particular situation where these N plane waves have the same amplitude. In this case we merely have $N_2N_6/N_4^2 = [15N^2 - 45N + 40]/9(N - 1)^2$ which increases regularly with increasing N, and is minimum for $N = 2$. In this last case $N_2N_6/N_4^2 = \frac{10}{9}$. It is interesting to remark that this conclusion is opposite to what one would obtain by considering the fourth order term alone in equation (9) and omitting the sixth order one. Since the fourth order term grows (compared to the second order one) with the number N plane waves, one would conclude that it is better to increase this number. The opposite turns out to be true because the sixth order term grows even faster with the number of plane waves. This shows quite clearly that, in contrast with what one might hope, the consideration of the fourth order term is not enough to conclude about the actual ground state of system.

We find then explicitely $A_0 = \frac{5}{36} + 2.02 \times 10^{-3}$. We note that this is quite close to the standard FFLO transition itself. For comparison the standard Clogston-Chandrasekhar [21,22] first order transition is given, as we have seen, by $A_0 = \frac{1}{8}$ so the difference with the standard FFLO transition is -1.39×10^{-2} which is also rather small. Since we know that the standard Clogston-Chandrasekhar transition and standard FFLO transition stay close beyond the vicinity of the TCP and that this proximity between these two lines extends down to zero temperature, it seems quite possible that the same is true for our first order transition line. This is indeed what has been found numerically by Matsuo *et al.* [14]. Hence it is very tempting to conclude from this proximity between first and second order lines that the first order transition is only very weakly first order. Naturally this point is very important experimentally since it would be quite difficult to distinguish between a second order transition and a very weak first order one.

On the other hand we know from the work of Burkhardt and Rainer [15] that the transition is second order in a 2D situation. So it is of some interest to consider formally an arbitrary dimension to see how one goes from the 2D to the 3D situation. For the case of our planar order parameter, there is no difficulty in working with an arbitrary dimension D. As we have mentionned at the end of Section 2, one has just to modify some coefficients in equation (7): here one has to multiply the Q^2 terms by $3/D$ and the Q^4 terms by $15/D(D + 2)$. Reproducing then the above analysis, we obtain that the FFLO transition occurs for $A_0 = \frac{D+2}{12D}$ with a wavevector $\bar{q}^2 = \frac{D+2}{6}$. One finds again that, in order to obtain a phase transition higher than FFLO, the coefficient of the second term in the r.h.s. of equation (21) has to be negative. But this coefficient is now in general $N_4(\frac{\alpha(D+2)}{D} - 1)/4$ with explicitely $\alpha = \frac{5}{9}$. So we find $D_c = 2.5$ as the critical dimension to have a first order transition. The result equation (21) for the location of the transition for this planar state becomes:

$$A_0 = \frac{D+2}{12D} + \frac{(2D-5)^2}{15D(7D-10)} \qquad (27)$$

with the value of $\bar{\Delta}_m$ of $\bar{\Delta}$ at the threshold given by:

$$\bar{\Delta}_m^2 = \frac{4}{5}\,\frac{2D-5}{7D-10}. \qquad (28)$$

Since we have found a critical dimension $D_c = 2.5$ which is halfway between the physical situations $D = 2$ and $D = 3$, from this point of view, the first order transition we have found is not 'very near to be second order', in contrast to what we suggested above. Naturally the best way to conclude on this point is to look at the value of the order parameter at the first order transition equation (28) which is $\bar{\Delta}_m \simeq 0.27$. Although this is clearly smaller than the corresponding value $\bar{\Delta}_m = 1$ for the uniform BCS state, the order of magnitude is similar so the first order transition is not very weak in this respect. This seems in agreement with results given in reference [14] where the jump of the order parameter is also sizeable.

In two dimensions we can again make use of equation (20) to compare all the possible states. As we have seen the transition is second order and we can neglect the sixth order term as in the original LO analysis. The fourth order term is positive and, roughly speaking, below the transition the state with smaller fourth order term is selected. This leads to take $N_4/N_2^2 = 3(2-S_2)/2$ as small as possible. So we find again that the single pair state $N = 2$ is selected, which corresponds to the state investigated by Burkhardt and Rainer [15].

In closing this section, let us note that we have have found a non- degenerate minimum for the free energy, namely the planar state $\cos(\mathbf{q}\cdot\mathbf{r})$ corresponding to two plane waves with opposite wavevectors and equal amplitude in equation (11). However by adding to this order parameter another cosine with a small amplitude, namely $\epsilon\cos(\mathbf{q_1}\cdot\mathbf{r})$, we can have a free energy arbitrarily close to the planar state free energy. Conversely the location for the transition can be made arbitrarily close to the planar state transition. These states are highly degenerate since the relative directions of $\mathbf{q}$ and $\mathbf{q_1}$ do not matter. These are some kind of excited states for our system with arbitrarily small energy and it is possible to speculate that they play a role in the physics of our system.

5 Beyond the LO subspace

In the preceding section we have restricted to the LO subspace our search for the state with lowest free energy at the transition and therefore with highest critical temperature. While this restriction is justified when the free energy expansion is limited to the fourth order term (this term being treated as a perturbation), it is no longer valid when the sixth order term is included. And indeed the actual minimum state does not belong to the LO subspace, since it can be checked that equation (11) can not satisfy the Euler-Lagrange equation corresponding to equation (10) because of the non linear terms. However we have found in the preceding section a non degenerate minimum corresponding to the planar state. The selection of the wavevectors of this state was due to the fourth order term while the number of components was controlled by the sixth order one, which is otherwise quite structureless. If the nonlinearities produce only a rather small change on the solution, it is reasonable to look for the actual solution as being 'near' the solution $\cos(\mathbf{q}\cdot\mathbf{r})$ we have found in Section 4. In particular it is reasonable to look for an order parameter with a one-dimensional dependence on $\mathbf{r}$ and we will also assume it to be real. And indeed we will find a solution which is quite near $\cos(\mathbf{q}\cdot\mathbf{r})$. Although this argumentation is only a self-consistent one (we can not exclude that a strong nonlinear modification of a three-dimensional solution produces the actual minimum), we note that the numerical exploration of Houzet *et al.* [13] has indeed produced a real one-dimensional order parameter for the minimum. We remark also that there is an analytical solution [23] in the case of one-dimensional space for this one-dimensional order parameter.

With a one-dimensional real order parameter the reduced free energy simplifies into:

$$F = \int \mathrm{d}x \left[A_0\bar{\Delta}^2 - \frac{1}{3}\bar{\Delta}'^2 + \frac{1}{5}\bar{\Delta}''^2\right] - \int \mathrm{d}x \left[\frac{1}{4}\bar{\Delta}^4 - \frac{5}{6}\bar{\Delta}^2\bar{\Delta}'^2\right] + \frac{1}{8}\int \mathrm{d}x\,\bar{\Delta}^6 \qquad (29)$$

where $\bar{\Delta}'$ and $\bar{\Delta}''$ are first and second derivative of $\bar{\Delta}$ with respect to x. Although we deal with a nonlinear problem we can still minimize with respect to the amplitude of the order parameter, just as we have done at the end of the preceding section (this works actually also for a three-dimensional order parameter). We set $\bar{\Delta}(x) = a\delta(x)$, where $\delta(x)$ is a normalized spatial function (for example by $\int \delta^2 = 1$). $\delta(x)$ gives the shape of the order parameter while a corresponds to its amplitude. When we substitute in equation (29) we obtain for F/a^2 a quadratic form in a^2. Writing again that, for a specific function $\delta(x)$, this form has double root at the transition leads us to an expression for A_0 which does not depend anymore on the normalization of δ:

$$A_0 = \frac{[\int \delta^4 - \frac{10}{3}(\delta\delta')^2]^2}{8\int\delta^2\int\delta^6} + \frac{\int \frac{1}{3}\delta'^2 - \frac{1}{5}\delta''^2}{\int\delta^2} \qquad (30)$$

and we want now to maximize A_0. Similarly the condition that the free energy is zero at the transition imply:

$$\int \bar{\Delta}^6 = \int \bar{\Delta}^4 - \frac{10}{3}\bar{\Delta}^2\bar{\Delta}'^2 \qquad (31)$$

or equivalently:

$$A_0\int \bar{\Delta}^2 = \int \frac{1}{8}\bar{\Delta}^6 + \frac{1}{3}\bar{\Delta}'^2 - \frac{1}{5}\bar{\Delta}''^2 \qquad (32)$$

which gives the amplitude of the order parameter.

The condition that A_0 is maximum can be obtained from equation (30) but it is easier to deduce it from equation (29). One finds the ordinary nonlinear differential equation:

$$\frac{1}{5}\bar{\Delta}'''' + \left(\frac{1}{3} - \frac{5}{6}\bar{\Delta}^2\right)\bar{\Delta}'' - \frac{5}{6}\bar{\Delta}\bar{\Delta}'^2 + \frac{3}{8}\bar{\Delta}^5 - \frac{1}{2}\bar{\Delta}^3 + A_0\bar{\Delta} = 0 \quad (33)$$

(analytically this equation can be integrated once and then reduced to a second order nonlinear differential equation but this is of no real help for numerics). One checks readily that the cosine form of the order parameter $\bar{\Delta}(x) = a\cos(\bar{q}x)$ is not a solution. It satisfies only the linear part of the equation, which leads to $A_0 = \bar{q}^2/3 - \bar{q}^4/5$. Looking for the maximum of A_0 gives the FFLO result $A_0 = 5/36 \simeq 0.1389$ with $\bar{q}^2_{FFLO} = 5/6$. This FFLO solution corresponds to keep only the second term in the r.h.s. of equation (30). This term produces for A_0 a kind of (inverted) effective potential $\bar{q}^2/3 - \bar{q}^4/5$ which has a strong maximum for the FFLO solution. When we write a Fourier expansion of the solution this potential selects quite effectively the wavevectors in the close vicinity of the FFLO result $\bar{q}_{FFLO}$.

Indeed the numerical exploration of equation (33) for A_0 very close to the maximum gives for the solutions one or two basis frequencies q_0 and q_1 which are close to $\bar{q}_{FFLO}$. The other frequencies appearing in a Fourier analysis are simply odd combinations of q_0 and q_1 like $2q_0 \pm q_1$, $2q_1 \pm q_0$, $3q_0$, $4q_1 \pm q_0$, etc. The weights of these frequencies depend on their order, *i.e.* the higher the frequency, the smaller the weight. This shows explicitely the strength of the effective FFLO potential in A_0 (the second term in the r.h.s. of Eq. (30)). We have investigated analytically the efficiency of a small frequency splitting by writing $q_1 = q_0 + \epsilon$ with $\epsilon \to 0$ (but $\epsilon \neq 0$ because this limit is singular). However a single frequency $q_1 = q_0$ turned out to be always better. We are finally lead to the conclusion that going beyond the LO subspace produces only small corrections due to nonlinearities. These corrections correspond to odd harmonics of the fundamental frequency $\bar{q}$ and they are small because of the efficiency of the effective FFLO potential. Actually if we assume from the start that the corrections to $\delta_0(x) = \cos(\bar{q}x)$ are small, we can write $\delta(x) = \delta_0(x) + \delta_1(x)$, with $\delta_1(x)$ small, and perform a first order expansion of equation (30). One finds readily that it is most favorable to take $\delta_1(x)$ proportional to $\cos^3(\bar{q}x)$ (another contribution from $\cos^5(\bar{q}x)$ is quite small), but one has to go to second order to find the amplitude. Actually, once it is proved that this harmonic expansion is the correct answer, it is much easier to minimize equation (30) numerically which avoids cumbersome calculations. Specifically we considered the trial function $\delta(x) = \cos(\bar{q}x) + a_3\cos(3\bar{q}x + \phi_1) + a_5\cos(5\bar{q} + \phi_2)$ and numerically maximized A_0 (some more complete trial functions that we also tried eventually reduced to this form when maximized). We found $a_3 = -1.33 \times 10^{-2}$, $a_5 = 1.62 \times 10^{-4}$, $\bar{q} = 0.793$, $\phi_{1,2} = 0$ and $A_0 = 0.141604$ (which is in full agreement with reference [13]). This form is very close to the cosine solution and the improvement in A_0 compared to our result equation (22) for the cosine solution $A_0 = 31/220 \simeq 0.14091$ is pretty small in absolute values, although it gives a significant increase of 7×10^{-4} to our gain of 2.02×10^{-3} compared to the FFLO result. Our result $\bar{q} = 0.793$ has to be compared to our result in LO subspace from equation (23) $\bar{q} = 0.829$ and to $\bar{q}_{FFLO} = 0.913$.

Let us finally indicate that, in contrast to the weak nonlinearities we have just found at the transition, one finds that higher order harmonics become increasingly important when one goes deeper into the superfluid, which corresponds to decrease A_0. This makes this regime beyond the scope of our study. Actually it has been found quite interestingly that, in 2D [15] and in 3D [13], one goes progressively to a lattice of solitons, which leads to a second order phase transition to the uniform BCS state deep in the superfluid phase.

6 Conclusion

In this paper we have explored analytically the nature of the transition to the FFLO superfluid phases in the vicinity of the tricritical point, where these phases begin to appear. This region is convenient for the analytical study we make because, in the vicinity of this point, one can make use of an expansion of the free energy up to sixth order, both in order parameter amplitude and in wavevector. Despite this simplification one has still a complex nonlinear problem to solve. We have first explored the minimization of this free energy within the LO subspace, made of arbitrary superpositions of plane waves. We have seen that the standard second order FFLO phase transition is unstable and that a first order transition occurs at higher temperature. Within this subspace we have shown that it is favorable to have a real order parameter. Then among these states we have shown that those with the smallest number of plane waves are preferred. This leads to retain only two plane waves, corresponding to an order parameter with a $\cos(\mathbf{q}_0 \cdot \mathbf{r})$ dependence, in agreement with preceding work [13]. Finally we have shown that, when releasing the constraint of working within the LO subspace, the order parameter at the transition is only very slightly modified by higher harmonics contributions and we have been able to ascribe this result to the strong selection of the wavevector caused by the second order terms of the free energy, corresponding physically to the standard FFLO transition.

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20. Actually this is only correct for the generic situation where the $\{\mathbf{q}_i\}$ have no other specific relation than being associated in pairs $(\mathbf{q}_i, -\mathbf{q}_i)$. A simple case where additional degeneracy occurs is the set of 8 wavectors $(\pm 1, \pm 1, \pm 1)/\sqrt{3}$, which has a cubic symmetry. In this case we find for example $N_4 = 216$ instead of 168. The same caveat is valid for equations (24, 25) below. However we will see quite generally that it is unfavorable to increase the number of plane waves in order to minimize the free energy. Since these exceptional cases imply a large number of plane waves, they should have higher free energy and we do not expect them to alter our final result
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